T. Ekholm – 27/5/08 – Overview of contact homology

(Y^{2n-1}, α) contact manifold : $\alpha \wedge (d\alpha)^{n-1} \neq 0$

$\xi = \ker \alpha$ contact structure

<u>Legendrian submfld</u> := $\Lambda \subset Y$ $(n-1)$-submanifold, $T\Lambda \subset \xi$

<u>Ex</u>: (X, ω), $Y = \partial X$ convex : v Liouville vector field $L_v \omega = \omega$

$\rightsquigarrow \alpha = i_v \omega$ contact on ∂X.

<u>Ex</u>: $Y = J^1(M) = T^*M \times \mathbb{R}$, $\alpha = dz - p\,dq$

$\Rightarrow$ the zero section is Legendrian

- symplectization: $(Y \times \mathbb{R}, d(e^t \alpha))$ symplectic $\supset \Lambda \times \mathbb{R}$ Lagrangian

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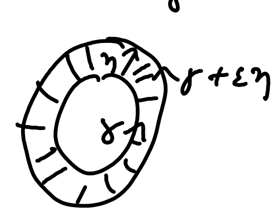
<u>Contact homology and SFT</u>:

$\gamma \mapsto \int_\gamma \alpha$ function on $\mathrm{Loop}(Y)$; do Morse theory on $\{\gamma \,/\, \int_\gamma \alpha \geq 0\}$

$$\int_{\gamma + \varepsilon\eta} \alpha = \int_\gamma \alpha + \varepsilon \int d\alpha(\dot\gamma, \eta) + O(\varepsilon^2)$$

(by Stokes)

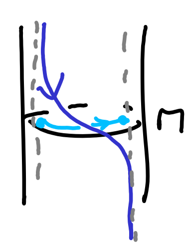


$\Rightarrow$ critical points $d\alpha(\dot\gamma, \cdot) = 0$

are orbits of the <u>Reeb vector field</u> $\begin{cases} d\alpha(R, \cdot) = 0 \\ \alpha(R) = 1. \end{cases}$

- Can do Morse theory by looking at solutions of

$(\dot x, \dot t) = (-\nabla f, -1)$ on $M \times \mathbb{R}$

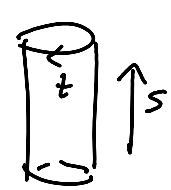


Here: Equip ξ with a complex structure $J : \xi \to \xi$

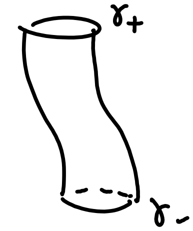
and extend to $Y \times \mathbb{R}$ by setting $J \partial_t = R$

and look at holom. curve equation $\dfrac{\partial u}{\partial s} + J \dfrac{\partial u}{\partial t} = 0$

for $u : \mathbb{R} \times S^1 \to Y \times \mathbb{R}$



The condition $J\partial_t = R \Rightarrow$ trivial cylinders on Reeb chords are J-holomorphic

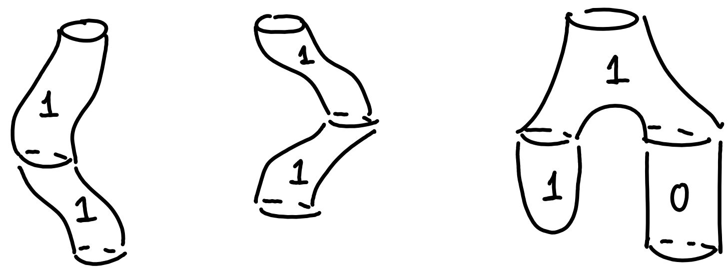
In order to define diff! in Morse-Floer theory, count 

mod $\mathbb{R}$-transl?

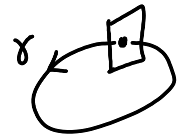
To study $d^2 \overset{?}{=} 0$, need to compactify moduli space

Thm: (Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder)

> The space of holom. curves admits a compactification consisting of multi-level curves



etc....

★ Grading: 

$d\phi_t$ Reeb flow linearization along the orbit γ

Note: $\exists$ complex trivialization Z

$\rightarrow CZ(\gamma) := \text{Maslov}(\text{graph}(d\phi_t))$

Conley-Zehnder index

Then set $\underline{|\gamma| := CZ(\gamma, Z) + (n-3)}$

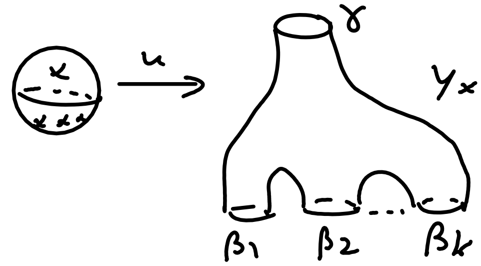
Def: A Reeb orbit γ^k is bad if $|\gamma^k|$ and $|\gamma|$ have different parities.

k^{th} iterate of a primitive orbit γ

Def: The DGA of Y: $Q(Y)$ = free unital supercomm. algebra generated by good Reeb orbits. (over $\mathbb{Q}[H_2(Y)]$)

$d: Q(Y) \to Q(Y)$ defined on generators (& linearity + Leibniz rule)

$$\mathcal{M}(\gamma, \beta_1 \dots \beta_k) = \left\{ \right.$$



$$Y \times \mathbb{R}, \quad du + J \cdot du \cdot j = 0 \left. \right\}$$

$$d\gamma = \sum_{\dim \mathcal{M}(\gamma; \beta_1 .. \beta_k) = 1} \frac{1}{k_{\beta_1} \dots k_{\beta_k}} \left| \mathcal{M}(\gamma; \beta_1 .. \beta_k)/\mathbb{R} \right| \beta_1 \dots \beta_k$$

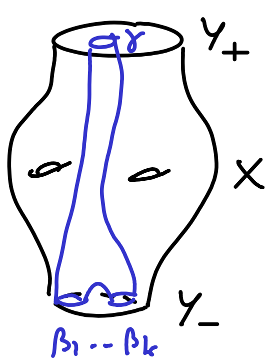
↳ multiplicities of the orbits $\beta_1 \dots \beta_k$

<u>Thm:</u> $d^2 = 0$

[proof should soon be finally complete]

<u>Rmk:</u>
- $d = d_0 + d_1 + d_2 + \dots$ according to # neg. punctures
  d_1 doesn't square to zero ... since d_0 = obstruction.
  → we'll focus on special case where ∃ linearized homology.
- [Getzler: can reformulate this as a <u>curved L_∞-algebra</u> with differential $= d_1$, bracket $= d_2$, ...]

- <u>SFT:</u>



$$\rightsquigarrow \quad \begin{array}{c} Q(Y_+) \\ \big\downarrow \Phi_X \\ Q(Y_-) \end{array}$$

$$\phi_X(\gamma) := \sum_{\mathcal{M}(\gamma; \beta_1 .. \beta_k) = 0} \frac{1}{\prod k_\beta} |\mathcal{M}| \, \beta_1 \dots \beta_k .$$ is a <u>chain map!</u>

<u>Thm:</u> $d^- \phi_X - \phi_X d^+ = 0.$

Thm: A deformation of the cobordism gives chain homotopic maps $\phi_0, \phi_1 : Q(Y^+) \to Q(Y^-)$, ie. $\exists$ a chain homotopy $K : Q(Y^+) \to Q(Y^-)$, $\left(\begin{array}{l} \gamma \text{ Reeb orbit generator of } Q(Y^+) \\ \qquad \mapsto K(\gamma) \in Q(Y^-) \end{array} \right)$

s.t. $\Phi_1 - \Phi_0 = d\,\Omega_K - \Omega_K\, d$ where

$$\Omega_K(\beta_1 \dots \beta_k) = \sum_j K(\beta_j)\, \overline{\Phi_{01}}(\beta_1 \dots \beta_{j-1}, \beta_{j+1} \dots \beta_k)$$

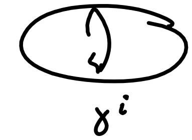
$$\overline{\Phi_{01}}(\beta_1 \dots \beta_k) = \frac{1}{2^k} \sum_{\sigma \in \{0,1\}^k} \Phi_{\sigma_1}(\beta_1) \dots \Phi_{\sigma_k}(\beta_k)$$

(mix of Φ_0's and Φ_1's)

This in particular gives invariance of the theory under changes in choices of $J, \alpha, \dots$ (by considering (J_t, α_t) on $\mathbb{R} \times Y$):

Corollary: $\ker d / \operatorname{im} d =: CH(Y)$ is invariant under deformations

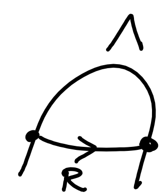
Ex: $S^{2n-1} = \left\{ \frac{|z_1|^2}{a_1} + \dots + \frac{|z_n|^2}{a_n} = 1 \right\}$ has $|\gamma^i| = 2i + (n-1)$



Relative version:

$\Lambda \subset Y$ Legendrian

$\rightsquigarrow A(Y, \Lambda) =$ $Q(Y)$-algebra freely generated by Reeb chords



$\partial : A(Y, \Lambda) \circlearrowleft$ defined by $\partial(\gamma) = d\gamma$ for γ closed orbit

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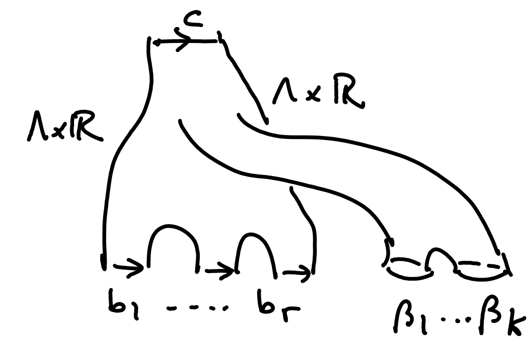
$$\partial c = \sum_{\dim \mathcal{M}(c;\beta_1 \dots \beta_k, b_1 \dots b_r) = 1} \frac{1}{\prod k_\beta} |\mathcal{M}| \, \beta_1 \dots \beta_k \, b_1 \dots b_r$$

counts



(NB: there are no bad chords to exclude ...) ← except in dim. 1 ....

- **Thm:** $\partial^2 = 0$

- **Thm:** cobordisms (X, L) give chain maps

  (X: symplectic, L: Lagrangian)

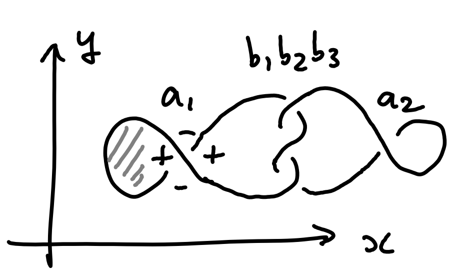
  $$\phi_{(X,L)} : A(Y^+, L^+) \to A(Y^-, L^-)$$

  and deformation of cobordisms give chain homotopies

as before, <u>except</u> noncomm. $\Rightarrow$ now for $K : A(Y^+, L^+) \to A(Y^-, L^-)$,

$$\begin{aligned} \Omega_K(b_1 \dots b_n) := {} & K(b_1)\, \phi_1(b_2) \dots \phi_1(b_n) \\ & + \phi_0(b_1)\, K(b_2)\, \phi_1(b_3) \dots \phi_1(b_n) \\ & + \dots + \phi_0(b_1) \dots \phi_0(b_{n-1})\, K(b_n). \end{aligned}$$

<u>Ex:</u> for $\dim \Lambda = 1$ in $Y = J^1(\mathbb{R}) = \mathbb{R}^3$: <u>Chekanov</u>



$$\partial a_1 = 1 + b_3 + b_1 + b_3 b_2 b_1$$
$$\partial a_2 = 1 + b_3 + b_1 + b_1 b_2 b_3$$
$$\partial b_1 = \partial b_2 = \partial b_3 = 0$$